# BROKERING MOVES TO SUPPORT ADVANCES IN A PRESERVICE ELEMENTARY TEACHER'S DISCOURSE ABOUT HIERARCHICAL GEOMETRIC RELATIONSHIPS

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*Limited literature addresses how elementary preservice teachers (PSTs) can advance their thinking about hierarchical geometric relationships. Informed by a commognitive perspective, we investigated this phenomenon as a matter of discursive change, focusing on word use, visual mediators, and narratives. We report on a teaching experiment involving a PST whom we call Mariah. Mariah's progress seemed to be driven primarily by metalevel changes in her word use and interpretations of diagrams. These changes supported shifts in her narratives regarding hierarchical geometric relationships. Instructor moves related to these discursive changes involved brokering between multiple discourse communities. Our analysis reveals nuances of communication that we do not see highlighted in the literature on PST education.*

Keywords: Communication, Geometry and Spatial Reasoning, Preservice Teacher Education

There is little literature regarding the thinking or learning of elementary preservice teachers (PSTs) concerning hierarchical geometric relationships (Browning et al., 2014). *Hierarchical geometric relationships* refer to set–subset relationships between different types of shapes based on their properties (Fujita & Jones, 2007). Some articles frame their findings in terms of deficits in PSTs' thinking (Fujita & Jones, 2007; Pickreign, 2007). Others report results of intervention studies but do not illuminate the learning process (Brunheira & da Ponte, 2019; Yi et al., 2020). The literature and our teaching experience have shown us that reasoning about and representing hierarchical geometric relationships can be challenging for PSTs. For this topic and many others, the field needs accounts of PSTs' thinking and learning that contribute insights and support theoretically grounded instructional sequences.

## Clues and Unanswered Questions in the Literature

Yi et al. (2020) conducted an intervention study and reported evidence that PSTs' geometry knowledge for teaching 2-D shapes can improve significantly over a three-week period. The description of their instructional sequence mentions one activity in which PSTs "used Venn diagrams to categorize [quadrilaterals] based on their similar and different properties" (p. 5) but does not elaborate. The assessment included items on quadrilateral properties and relationships (among others) but did not address some challenging aspects (e.g., categorization of trapezoids). Thus, Yi et al. offer some evidence of improvement, but their account does not explain PSTs' difficulties with the topic of hierarchical relationships or how those difficulties can be overcome.

Brunheira and da Ponte (2019) report on a teaching experiment focused on classification of quadrilaterals and prisms. They offer examples of a PST's work that involves what seem to us two contrasting interpretations of Venn diagrams (p. 76), rather than a shift in thinking about the hierarchical relationships among the shapes. This distinction is left unexplored. The authors report progress in PSTs' reasoning but also lingering difficulties, and they suggest the need to explore "other factors" such as "language interpretation and logical reasoning" (p. 80).

Lamberg, T., & Moss, D. (2023). *Proceedings of the forty-fifth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1). University of Nevada, Reno.

Overall, the literature leaves us with unanswered questions. It does not provide insights into PSTs' geometric thinking, nor does it document viable learning processes for specific topics (Browning et al., 2014). We believe a key to understanding PSTs' thinking and supporting their learning related to hierarchical geometric relationships lies in those aforementioned "other factors" that we have not seen deeply explored in the literature to date.

## Theoretical Framework

This study is primarily informed by a commognitive perspective, according to which thinking and communicating are one and the same. Through this lens, learning takes the form of discursive change. In the case of an individual PST, we can characterize aspects of their mathematical discourse at points in time, and thus document changes that took place in their discourse. These changes are tantamount to changes in the person's thinking (Sfard, 2007).

We focus here on word use, visual mediators, and narratives as properties of discourse. Mathematical discourse involves specifically mathematical vocabulary, along with everyday words: "Although shape- and number-related words may appear in nonspecialized, colloquial discourses, literate mathematical discourses as practiced in schools or in academia dictate their own more disciplined uses of these words" (Sfard, 2007, p. 571). Mathematical discourse often involves symbolic artifacts that serve important roles as *visual mediators* of mathematical communication. In our case, Venn diagrams are a symbolic artifact of particular interest. *Narratives* are statements that are "subject to *endorsement* or rejection, that is, to being labeled as *true* or *false*" (p. 572). Narratives regarding properties of quadrilaterals and relationships based on those properties are central to the topic of hierarchical geometric relationships.

We draw on related literature to characterize the role of the instructor in our study as a broker between discourse communities: "By definition, a broker is someone who can facilitate communication and fluidity of practices between different communities and who has membership status in all the different communities" (Zandieh et al., 2017, p. 97). In the case of Zandieh et al., the communities of interest were the local classroom community and the broader mathematical community. Their analysis examined ways in which an instructor helped to link student practices with elements of practice from the broader mathematical community. This connection to a "broader mathematical community" relates to Sfard's (2007) notion of a *leading discourse.* The leading discourse has special status relative to individual discourses. At the same time, the broker as interpreter is concerned with honoring students' thinking so that "interpreting between communities facilitates the students' sense of ownership of ideas" (p. 98).

In keeping with the view of learning as discursive change, we are interested in how an instructor of elementary PSTs supports that change process. The notion of brokering moves is consistent with the participationist nature of Sfard's (2007) framework, and we have found it to be a useful way of framing the instructor's role. In the case of our study, there are four types of discourse communities involved: the broader mathematical community and the state standards, both of which are leading discourses; the non-leading discourse of other PSTs and "people" who were mentioned by the instructor but not present in the sessions; the collective discourse between Mariah and the instructor; and Mariah's individual geometric discourse.

According to Sfard (2007), studies guided by the commognitive framework should be able to address the following questions:

1. Focus on the object of learning: In the case under study, what kind of change was supposed to occur as a result of learning?

2. Focus on the process: How did the students and teacher work toward this change?
3. Focus on the outcome: Has the expected change occurred? (p. 566) We address each of these points in our Method and Results sections below.

## Method

We conducted an individual teaching experiment with Mariah (pseudonym). She was 18 years old and identified as a Black, Caribbean-American woman. She had been accepted into an Elementary Education program at a large research university in the Southeastern United States. The sessions took place in the summer before she began the program. The instructor (and first author) was 44 years old and identified as a White male. The teaching experiment spanned three one-hour sessions. These were conducted via Zoom and involved Google Slides and the Desmos Geometry Tool. The learning goal was for Mariah to engage in a geometric discourse based on official definitions, emphasizing set–subset relationships, and involving the use of diagrams to represent those relationships. The culminating task was to construct "one big diagram" involving several types of quadrilaterals and to explain the relationships between them.

In Session 1, Mariah was asked about her meaning for the term *Venn diagram* and her interpretations of various diagrams. She was asked to generate examples of diagrams to represent relationships, both mathematical and non-mathematical. Mathematical topics of discussion included the square–rectangle and parallelogram–rhombus relationships. In Session 2, the instructor introduced the distinction between inclusive and exclusive definitions. Mariah was invited to think about the implications of definitions. She discussed various relationships and associated diagrams involving sets and subsets. She worked to create a diagram to represent the rectangle–rhombus–square relationship. In Session 3, Mariah explored trapezoids and considered both inclusive and exclusive definitions for *trapezoid*. Mariah worked to produce "one big diagram" to represent the hierarchical relationships between quadrilaterals, parallelograms, rectangles, rhombi, squares, and trapezoids.

The instructional sequence was extracted from a geometry unit that the instructor had taught in a course for PSTs the previous semester. The summer teaching experiment with Mariah provided the opportunity to closely study the thinking and learning of an individual PST through the lens of discursive change. In this context, we asked the following research questions:

1. How did Mariah's discourse related to hierarchical geometric relationships change over the course of the teaching experiment?
2. What brokering moves did the instructor use to support Mariah in shifting her discourse?

Videos of the Zoom sessions were transcribed for analysis. The analysis in answer to Research Question 1 was informed by the commognitive framework, together with previous experience teaching the topic of hierarchical geometric relationships and initial investigations of PSTs' thinking related to this mathematical topic (Whitacre & Caro-Rora, 2022). Our analysis of Mariah's discourse focused on the following aspects: (a) her word use, especially concerning diagrams, definitions, and hierarchical relationships; (b) her interpretation and use of symbolic artifacts, especially Venn diagrams; and (c) her narratives concerning quadrilateral properties and relationships. We identified patterns in the data to produce summary characterizations of details of Mariah's discourse in each of the sessions. We made chronological comparisons within and across sessions to identify shifts in aspects of Mariah's discourse.

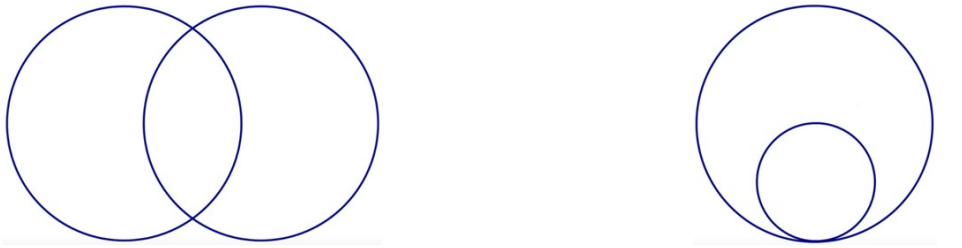
To answer Research Question 2, we analyzed the same data set with a focus on the instructor's utterances. Building on the answers to Research Question 1, we investigated the interactions that preceded and appeared to facilitate or support the observed shifts in Mariah's discourse. This analysis is informed by the notion of brokering (Rasmussen et al., 2009; Zandieh et al., 2017) because the instructor's role was marked by a focus on communication and clarification. Through our ongoing analyses, we continue to refine these characterizations of the interlocutors' interactions and the process of change that took place across the sessions.

## Results

In answer to Research Question 1, we highlight changes in Mariah's discourse in terms of word use, visual mediators, and narratives across the three sessions of the teaching experiment. We present characterizations of her discourse and changes in her discourse in chronological order. In answer to Research Question 2, we highlight the instructor's brokering moves and relate them to shifts in Mariah's discourse.

### Shifts in Mariah's Discourse

**Session 1.** Mariah initially indicated that a "Venn diagram" was for representing similarities and differences (i.e., she used the term Venn diagram to indicate a diagram that consisted of overlapping circles and was used for the purpose of comparing and contrasting). In Session 1, she began distinguishing between two types/interpretations of diagrams, which she named "compare and contrast diagrams" and "container diagrams" (Figure 1). She also introduced the language of "genre and subgenre" to describe the set and subset depicted in a "container diagram." Mariah initially talked about different arrangements of circles as determining the type of diagram: (a) overlapping circles were for showing similarities and differences, as in a "compare and contrast diagram," whereas (b) a circle within a circle showed a "genre and subgenre" relationship, as in a "container diagram."



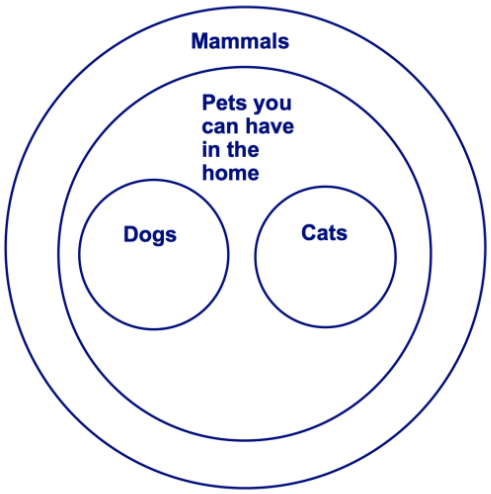
"Compare and contrast diagram" "Container diagram"

**Figure 1: Mariah's initial interpretations of Venn diagrams**

The properties that Mariah voiced, together with her answers to "Is it still?" questions (e.g., "If we move this vertex of the parallelogram construction until it becomes a rectangle, is it still a parallelogram?"), constituted her working definitions of specific quadrilaterals as of Session 1: Mariah talked about rectangles and parallelograms inclusively regarding their side lengths (e.g., she endorsed the narrative that a square is a "subgenre" of rectangle). At the same time, she talked about parallelograms exclusively regarding their angles (i.e., she endorsed the narrative that a rectangle was *not* a type of parallelogram).

**Session 2.** Mariah became aware of the terms *inclusive* and *exclusive* as applied to definitions of geometric shapes. She was able to distinguish between the two types of definitions and relate them to diagrams when asked to do so, but she rarely used these terms spontaneously.

Mariah recognized that for container diagrams, different arrangements were possible (beyond the circle-in-circle archetype in Figure 1), and these indicated the nature of various possible "genre and subgenre" relationships. She distinguished different arrangements of circles from types/interpretations of diagrams. Regarding container diagrams, overlap took on a different meaning in Mariah's discourse: "subgenre of both" (Figure 2b). Similarities were shown in membership in a larger set (Figure 2a), rather than in an overlapping region. She indicated that she did not think it would make sense for the "Dogs" and "Cats" ovals in Figure 2a to overlap.

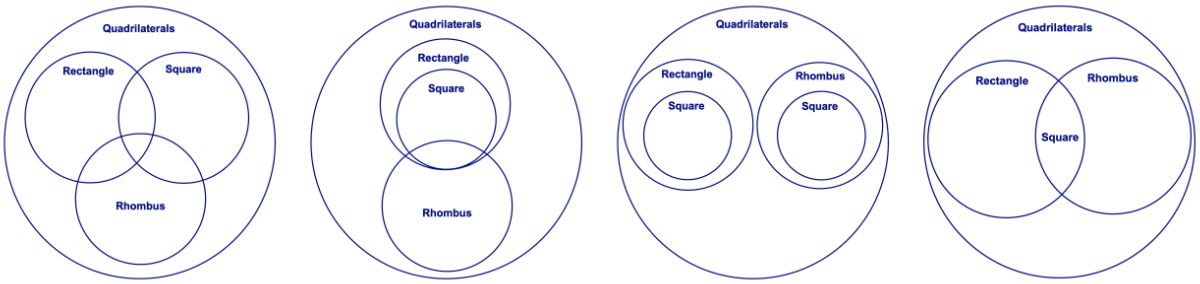



2a. Mariah's "container diagram" depicting similarities between dogs and cats as membership in common "genres"


2b. Mariah's "container diagram" using overlap to show that mint chocolate chip ice cream was a "subgenre" of both mint ice cream and chocolate chip ice cream

**Figure 2: Progress in Mariah's use of "container diagrams"**

When asked to create a container diagram for rectangles, rhombi, and squares, Mariah worked through four versions (Figure 3). In the first two, she spoke about rhombi as exclusive regarding angles. Given an inclusive definition of rhombus, she then described a square as a "subgenre of both" rectangle and rhombus, eventually settling on the fourth version (far right).




Mariah's first version, which she spontaneously abandoned

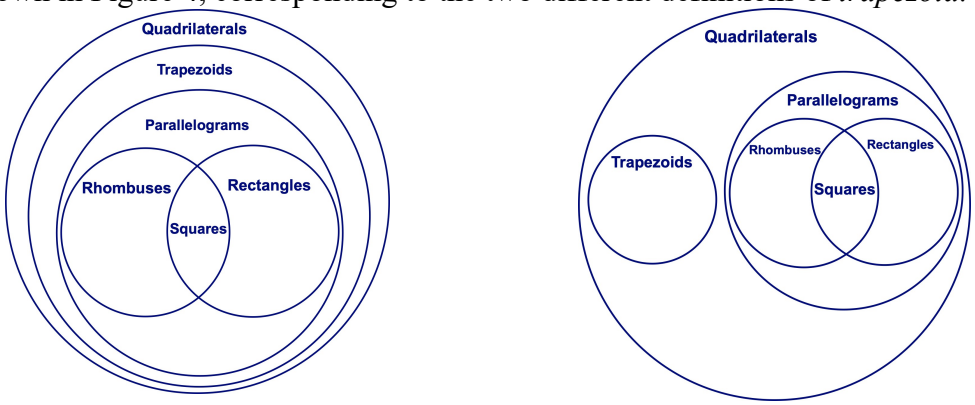
Mariah's second version, revised in response to questions about the regions

Mariah's third version, based on an inclusive definition of rhombus

Mariah's final diagram, inspired by other overlapping "container diagrams"

**Figure 3: Mariah's "container" diagrams for the rectangle–rhombus–square relationship**

**Session 3.** In this session, "inclusive" and "exclusive" became part of Mariah's individual discourse—not only terms she understood, but terms that she used repeatedly and spontaneously. Mariah also fluently used "container diagrams" to represent hierarchical relationships, including nested and overlapping relationships. She was eventually able to create diagrams to represent how several types of quadrilaterals were related, to explain each of those relationships based on definitions, and to explain how those relationships were represented in her diagrams. Mariah spoke about trapezoids exclusively. However, she was able to construct the two "container diagrams" shown in Figure 4, corresponding to the two different definitions of *trapezoid.*




Treating Trapezoids Inclusively

Treating Trapezoids Exclusively

**Figure 4: Mariah's "container" diagrams relating several types of quadrilaterals Brokering Moves**

The instructor did not impose narratives regarding hierarchical relationships. He alternated between acting as an interviewer inquiring about Mariah's thinking and acting as a broker informing Mariah about other discourses that would not otherwise have been available during their interactions (e.g., sharing information about standards). We identified four specific subtypes of brokering moves that appeared to support Mariah in shifting her discourse:

**Sharing information about a non-leading, or peer, discourse.** To motivate the need for discussion of types/interpretations of diagrams, the instructor began by sharing his observation

(from courses for PSTs) that people often use and interpret diagrams and the term *Venn diagram* in different ways. This introduction set the stage for Mariah to share her interpretations and helped establish the need for these kinds of clarifications. Similarly, the instructor shared that people use "can be" and "can't be" in different ways when discussing geometric relationships; thus, it was important for people to make explicit what they meant by "can be" when discussing such relationships. Mariah clarified that she meant "can be considered" and subsequently used specific phrasing more consistently (e.g., "a square can be considered a rhombus").

**Promoting and valuing individual contributions to the collective discourse.** Once it was clear that Mariah herself had varying interpretations of diagrams, it became important to use terms to name types/interpretations of diagrams in the interest of clear communication. The instructor invited Mariah to suggest names, and she introduced "compare and contrast diagram," "container diagram," and "genre and subgenre." Mariah and the instructor consistently used Mariah's language from that point on, which enabled clear communication. It also signaled that Mariah's contributions were valued by making them part of the collective discourse.

**Sharing technical terminology from a leading discourse.** The instructor introduced terminology to name relevant distinctions that arose in Mariah's talk about quadrilaterals, especially the terms *inclusive* and *exclusive definitions*. Mariah subsequently distinguished between these two types of definitions and considered the implications of choices of definition for hierarchical relationships between quadrilaterals.

**Sharing meta-rules from a leading discourse.** The instructor shared information about the state's standards, especially the fact that the glossary used inclusive definitions for quadrilaterals, including for *trapezoid*. Thus, it was highly relevant for Mariah as a PST to become familiar and comfortable with those definitions. Mariah transitioned from classifying quadrilaterals based on her implicit, personal definitions to doing so based on explicit, official definitions.

## Discussion

Our learning goal for the instructional unit was achieved: There were noteworthy changes in Mariah's geometric discourse, culminating in her ability to construct "one big diagram" relating all the types of quadrilaterals that were discussed, based on their explicit definitions, and to explain the relationships shown in her diagram. Given the reported difficulties that PSTs have with this topic (Fujita & Jones, 2007; Pickreign, 2007), and the challenges of communication arising from different ways of using and interpreting language and diagrams (Whitacre & Caro-Rora, 2022), this outcome is a feat. Furthermore, our findings provide insights into Mariah's learning process: (a) the major changes in Mariah's discourse were meta-level changes; (b) these changes were facilitated by the instructor's brokering moves, which primarily involved interpreting between communities; and (c) progress in Mariah's reasoning about hierarchical geometric relationships was driven primarily by changes in her word use and interpretation of diagrams, not by changes in the logic of her narratives themselves.

### Meta-level Changes

We find that the major shifts identified in Mariah's discourse were meta-level discursive changes, meaning that they happened at the level of discourse about discourse (Sfard, 2007). These were primarily shifts in word use and interpretation of symbolic artifacts: (a) from "Venn diagram" meaning overlapping circles with a compare-and-contrast interpretation to "compare and contrast diagram" vs. "container diagram," (b) from associating overlap in circle diagrams with similarities under the compare-and-contrast interpretation to also being able to interpret overlap in a container diagram using the "genre and subgenre" interpretation (i.e., "a subgenre of

both"), and (c) from thinking/talking about each quadrilateral in her individual manner (e.g., treating parallelograms as inclusive in terms of side lengths but exclusive in terms of angles) to thinking/talking explicitly about inclusive vs. exclusive definitions and considering the implications of definitions on hierarchical relationships—better connecting her individual discourse to the leading discourse on quadrilateral properties and relationships.

**Teacher Educator as Broker**

The instructor made several brokering moves to support Mariah in transforming her discourse. These moves involved interpreting between discourse communities by (a) sharing information about a non-leading, or peer, discourse; (b) promoting and valuing individual contributions to the collective discourse; (c) sharing technical terminology from a leading discourse; and (d) sharing meta-rules from a leading discourse.

We speak in terms of *a* leading discourse because there are at least two leading discourses that were relevant in our teaching experiment. As in the case of Zandieh et al. (2017), the broader mathematical community was one of these. For example, the instructor, a mathematics teacher educator, shared the fact that there is not a consensus definition of *trapezoid* in mathematics textbooks or other authoritative sources. In this case, the leading discourse was the broader mathematical community. However, another leading discourse at play in the teaching experiment was determined by the state standards. Because Mariah was a preservice teacher, the standards were highly relevant and justifiably treated as authoritative by the interlocutors. This dimension may not arise in the case of undergraduate students studying linear algebra, as in Zandieh et al.

## Contributions to the Literature on Elementary PSTs and Geometry

This study offers contributions to the literature regarding PSTs' geometric thinking and learning. First and foremost, given a thoughtfully designed instructional sequence and support in the form of the instructor making key brokering moves, Mariah was able to transform her discourse in ways indicative of substantial progress. In our view, her progress toward the learning goal did not involve shifts from incorrect ideas to correct ones; instead, the process involved making distinctions to facilitate communication. The available literature on the topic of PSTs' geometric thinking is limited and may have thus far ignored nuances of word use and the roles of symbolic artifacts such as Venn diagrams as visual mediators in the learning process.

Responding to the call from Brunheira and da Ponte (2019) to investigate "other factors" that might explain PSTs' difficulties with this topic, such as "language interpretation and logical reasoning" (p. 80), our findings indicate that attention to language interpretation has the potential to support PSTs' learning in this area. Logical reasoning did not seem to be an issue for Mariah.

Mariah's successful learning process, together with our additional teaching experiments and investigations into PSTs' geometric thinking, indicate that the instructional sequence that we employed is potentially viable. A noteworthy feature of the instructional sequence is its explicitness about communication and the need to share interpretations, discuss terminology and facts about leading discourses, as well as to adopt local terminology (e.g., "container diagram" and "genre and subgenre") that functions as if shared in the classroom community (Rasmussen & Stephan, 2008). Our findings also inform revisions to the instructional sequence. Most notably, it was crucial to Mariah's progress in Session 2 to work with an explicit and official definition of *rhombus*. This and other iterations of the teaching experiment have highlighted advantages to earlier introduction of official definitions, after exploration of quadrilateral constructions and discussion of properties.

## Limitations and Conclusion

Naturally, we do not regard Mariah as representative of all elementary PSTs. In our ongoing research, we are analyzing data from another iteration of a teaching experiment on this topic, involving a pair of PSTs. We have also collected data from two sections of a course for PSTs taught in Spring Semester 2023. We have observed consistent patterns in PSTs' thinking about the term *Venn diagram* and in the need to make the kinds of distinctions regarding language and interpretations that were made in the teaching experiment with Mariah.

We reiterate the point that the field needs accounts of PSTs' mathematical thinking and learning that contribute insights and support theoretically grounded instructional sequences. In the case of hierarchical geometric relationships, we have made substantial progress in that regard. We have also found Sfard's (2007) framework to be useful for the purpose.

Our findings relate to the conference theme of *Engaging All Learners* in at least three ways: First, as noted in answer to Research Question 2, the instructor valued Mariah's language and contributions, so that the teacher–student relationship was not one directional. By adopting Mariah's terms (e.g., "container diagram") and making them part of the collective discourse, he supported changes in her discourse without devaluing her ideas. Second, the meta-level focus of the instructional sequence (i.e., consisting of discussion of how people use and interpret terms and diagrams) is inclusive of multiple perspectives and does not treat them as correct vs. incorrect. Instead, it emphasizes communication and the need to "speak the same language" across multiple discourse communities, as an approach to supporting learning. Third, the instructor modeled these practices in his interactions with Mariah, who is a PST and stands to benefit from examples of how to teach mathematics in ways that honor students' ideas, including their use of language and symbolic artifacts. These kinds of instructional experiences can help prepare PSTs to teach mathematics in ways that engage all learners in their future classrooms.

## Acknowledgments

We thank Mariah for her participation in the teaching experiment sessions.

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